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Acoustic streaming about a cylinder in orthogonal beams

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When orthogonal, plane sound waves of the same frequency, wavelength and amplitude, but with phase difference $\frac{1}{2}\pi$, are incident upon a circular cylinder there is a time-independent streaming about the cylinder which is in the form of a potential vortex separated from the cylinder by a thin, viscous Stokes layer. As the amplitude of the waves in one of the beams decreases, an additional viscous boundary layer is involved until, when the amplitude is sufficiently small, this flow structure is destroyed as fluid erupts from the boundary layer. In the limit of a single beam it is known that this eruption results in opposing jets perpendicular to the wavefronts of the oncoming wave.

1. Introduction

In this paper we consider the acoustic, or steady, streaming about a circular cylinder when plane sound waves are incident upon it from orthogonal directions. The frequency is the same for the waves in each beam as is the wavelength, assumed large when compared with the cylinder radius. The phase of the waves differs by $\frac{1}{2}\pi$, and the amplitude by a factor of λ. The ensuing motion is the same, when the reference frame is chosen appropriately, as if the cylinder performed an orbital motion in a fluid otherwise at rest. That is, the centre of the cylinder moves along an elliptical path, with λ the axis ratio, but its orientation remains fixed.

We assume that the velocity amplitude of fluid particles in the basic oscillatory motion is small compared with the cylinder radius. With this assumption the Reynolds stresses, which are responsible for the streaming motion, act within a thin Stokes layer adjacent to the boundary. This motion persists at the edge of the Stokes layer to drive a steady streaming flow outside it. The vorticity transport equation in the outer region involves a balance between diffusion of vorticity and convection by the Lagrangian mean velocity. A streaming Reynolds number, first identified by Stuart (1963) in the context of oscillatory flow problems, characterizes this outer flow. We discuss the outer flow in the limit of large streaming Reynolds number, in which case it assumes the form of a viscous boundary layer within a potential vortex flow. The circulation associated with the potential vortex is determined either analytically or numerically for a range of values of λ. There is a lower value of λ, λ_c, below which this model of the flow is no longer valid. Whereas for $\lambda > \lambda_c$ fluid particles progress in closed loops around the cylinder, for $\lambda \leqslant \lambda_c$ there is an eruption of fluid from the outer boundary layer to form jet-like structures that are often associated with these oscillatory motions.

The study is set within the context of earlier work. In particular we note that for $\lambda = 1$, Riley (1971) has shown that the outer boundary layer is not present and the Stokes-layer solution matches directly with the potential vortex solution. Whilst for

$\lambda = 0$ the symmetry of the flow implies that the circulation in the potential vortex must vanish. In that case diametrically opposed jets emerge from the outer boundary layer as predicted by Stuart (1966), and visualized by Davidson & Riley (1972). A particular feature of the flow under discussion is that, for $\lambda > \lambda_c$, although the cylinder experiences no net force, it does experience a torque which varies linearly with λ.

In §2 we pose the problem, whilst §3 is devoted to the solution procedure. The results are presented and discussed in §4.

2. Governing equations

As outlined in §1, we consider the flow induced about a circular cylinder, radius a, placed in orthogonal beams of plane sound waves. The waves in each beam have the same frequency, ω, and wavenumber, k, but differ in phase by $\frac{1}{2}\pi$. The velocity amplitude of the waves incident from the direction $\theta = 0$, see figure 1, is U_0, whilst that of the waves incident from $\theta = \frac{1}{2}\pi$ is λU_0, $0 \leqslant \lambda \leqslant 1$. We assume that the wavelength of the oncoming waves is large compared with the cylinder radius, so that $ka \ll 1$.

With a, U_0 and ω^{-1} as, respectively, a typical length, velocity and time with which to make our flow quantities dimensionless, the non-dimensional stream function ψ satisfies the equation

$$\frac{\partial}{\partial t}(\nabla^2 \psi) - \frac{\epsilon}{r}\frac{\partial(\psi, \nabla^2 \psi)}{\partial(r, \theta)} = \frac{\epsilon^2}{R_s}\nabla^4 \psi, \tag{1}$$

with the velocity components given by $v_r = r^{-1}\,\partial\psi/\partial\theta$, $v_\theta = -\partial\psi/\partial r$. We see from (1) that our flows are characterized by the two dimensionless parameters, $\epsilon = U_0/\omega a$ and $R_s = U_0^2/\omega\nu = \epsilon U_0 a/\nu$. We shall assume throughout that $\epsilon \ll 1$, and develop our solution accordingly, whilst $R_s = O(1)$. We note that R_s is a Reynolds number based upon the velocity ϵU_0, and we shall see that it plays such a role in association with the acoustic, or steady, streaming, as first anticipated by Stuart (1963). The boundary conditions which must be satisfied by ψ are the no-slip condition

$$\psi = \frac{\partial \psi}{\partial r} = 0 \quad \text{at} \quad r = 1, \tag{2a}$$

together with

$$\psi \sim -\mathrm{i}r\{(\lambda - 1)\cos\theta + \mathrm{e}^{\mathrm{i}\theta}\}\,\mathrm{e}^{\mathrm{i}t} \quad \text{as} \quad r \to \infty, \tag{2b}$$

where here, and throughout, the real part of any complex quantity is to be understood.

3. Solution procedure

With $\epsilon \ll 1$ we develop a solution for all flow variables in powers of ϵ. Thus, for example, we expand the stream function as

$$\psi(r, \theta, t) = \psi_0(r, \theta, t) + \epsilon\{\psi_1^{(s)}(r, \theta) + \psi_1^{(u)}(r, \theta, t)\} + O(\epsilon^2), \tag{3}$$

where, in anticipation of the steady streaming at $O(\epsilon)$, we have decomposed ψ_1 into a time-independent part, denoted by superscript 's', and a time-dependent part denoted by superscript 'u'.

If we substitute (3) into (1), then at leading order we have $\nabla^2 \psi_0 = 0$ the solution of which, subject to $(2b)$ and the first of $(2a)$, is

$$\psi_0 = -\mathrm{i}\left(r - \frac{1}{r}\right)\{(\lambda - 1)\cos\theta + \mathrm{e}^{\mathrm{i}\theta}\}\,\mathrm{e}^{\mathrm{i}t}. \tag{4}$$

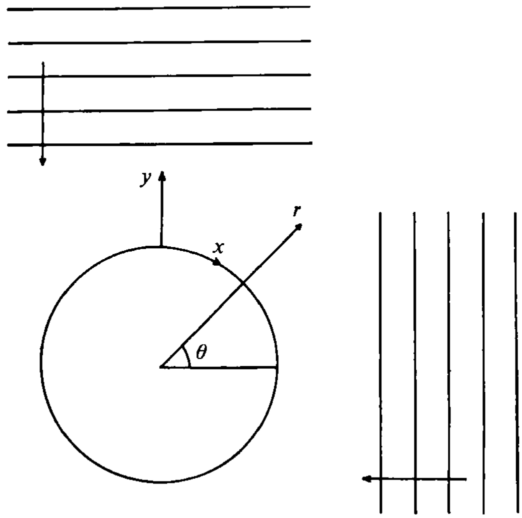

FIGURE 1. Definition sketch.

This solution is deficient, insofar as it does not satisfy the no-slip condition at $r = 1$, where

$$v_\theta = 2\mathrm{i}\{(\lambda-1)\cos\theta+\mathrm{e}^{\mathrm{i}\theta}\}\,\mathrm{e}^{\mathrm{i}t}, \tag{5}$$

is the predicted velocity of slip.

The slip velocity is accommodated within the classical Stokes shear-wave layer. At this point we choose, in conformity with earlier work (Riley 1965, 1975), to introduce the variables $x = \frac{1}{2}\pi-\theta$, $y = r-1$, as in figure 1. The Stokes-layer variables (see, for example, Riley 1967), are

$$\tilde{\psi} = \left(\frac{R_s}{2}\right)^{\frac{1}{2}}\frac{\psi}{\epsilon}, \quad \eta = \left(\frac{R_s}{2}\right)^{\frac{1}{2}}\frac{y}{\epsilon}, \tag{6}$$

in terms of which, equation (1) becomes, retaining terms up to and including $O(\epsilon)$,

$$\frac{\partial}{\partial t}\left(\frac{\partial^2\tilde{\psi}}{\partial\eta^2}\right)-\epsilon\frac{\partial(\tilde{\psi},\partial^2\tilde{\psi}/\partial\eta^2)}{\partial(x,\eta)} = \frac{1}{2}\frac{\partial^4\tilde{\psi}}{\partial\eta^4}. \tag{7}$$

In terms of these variables the velocity components are given by $\tilde{u} = \partial\tilde{\psi}/\partial\eta$ $(= -v_\theta)$, $\tilde{v} = -\partial\tilde{\psi}/\partial x$. As in the outer region we expand the flow variables in powers of ϵ so that, for example,

$$\tilde{\psi}(x,\eta,t) = \tilde{\psi}_0(x,\eta,t)+\epsilon\{\tilde{\psi}_1^{(s)}(x,\eta)+\tilde{\psi}_1^{(u)}(x,\eta,t)\}+O(\epsilon^2). \tag{8}$$

At leading order, the solution which satisfies the no-slip condition at $\eta = 0$, and matches with the slip velocity (5), is

$$\left.\begin{aligned}\tilde{\psi}_0 &= U(x)[\eta-\tfrac{1}{2}(1-\mathrm{i})\{1-\mathrm{e}^{-(1+\mathrm{i})\eta}\}]\,\mathrm{e}^{\mathrm{i}t},\\ \text{where}\qquad U(x) &= -2\mathrm{i}\{(\lambda-1)\sin x+\mathrm{i}\mathrm{e}^{-\mathrm{i}x}\}.\end{aligned}\right\} \tag{9}$$

If now, in (7) with $\tilde{\psi}$ as in (8), we consider the terms $O(\epsilon)$, and take a time average denoted by $\langle\cdot\rangle$ then since $\langle\tilde{\psi}_1^{(u)}\rangle = 0$ we have

$$-\frac{1}{2}\frac{\partial^4\tilde{\psi}_1^{(s)}}{\partial\eta^4} = \left\langle\frac{\partial(\tilde{\psi}_0,\partial^2\tilde{\psi}_0/\partial\eta^2)}{\partial(x,\eta)}\right\rangle. \tag{10}$$

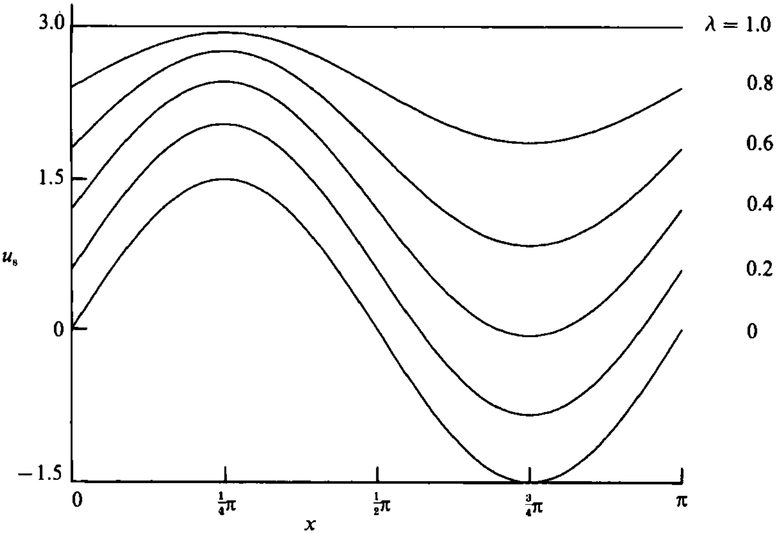

FIGURE 2. The variation of the slip velocity u_s, equation (13), for various values of λ. Note that $u_s(x+\pi) = u_s$.

The appropriate solution of (10) that satisfies the no-slip condition at $\eta = 0$, with $\tilde{u}_1^{(s)}$ bounded as $\eta \to \infty$, gives

$$\tilde{u}_1^{(s)} = \frac{\partial \tilde{\psi}_1^{(s)}}{\partial \eta} = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}x}(UU^*)\,(\mathrm{e}^{-\eta}\sin\eta + \tfrac{1}{4}\mathrm{e}^{-2\eta} - \tfrac{1}{4})$$
$$-\mathrm{Re}\left[U^*\frac{\mathrm{d}U}{\mathrm{d}x}\{\tfrac{1}{2}(1+\mathrm{i})\,\eta\mathrm{e}^{-(1-\mathrm{i})\eta} + \mathrm{i}\mathrm{e}^{-(1-\mathrm{i})\eta} - \tfrac{1}{2}\mathrm{e}^{-(1-\mathrm{i})\eta} - \tfrac{1}{4}\mathrm{i}\mathrm{e}^{-2\eta} - \tfrac{3}{4}\mathrm{i} + \tfrac{1}{2}\}\right], \quad (11)$$

where $U(x)$ is as in (9), and an asterisk denotes the complex conjugate. From (11) the shear stress contribution at the cylinder surface is given by

$$(\tfrac{1}{2}R_s)^{\frac{1}{2}}\frac{\partial \tilde{u}_1^{(s)}}{\partial \eta}\bigg|_{\eta=0} = \frac{R_s^{\frac{1}{2}}}{2\sqrt{2}}\left\{\mathrm{Re}\left(U\frac{\mathrm{d}U^*}{\mathrm{d}x}\right) + \mathrm{Im}\left(U\frac{\mathrm{d}U^*}{\mathrm{d}x}\right)\right\}$$
$$= (\tfrac{1}{2}R_s)^{\frac{1}{2}}\{(1-\lambda^2)\sin 2x + 2\lambda\}. \quad (12)$$

From expression (12), we may deduce that there is no net force on the cylinder, but that it experiences a clockwise torque which varies linearly with λ.

To complete the solution to $O(\epsilon)$ we must return to the outer solution. First, however, note that from (11) the limit process $\eta \to \infty$ predicts a steady streaming velocity at the edge of the Stokes layer as

$$u_s = \tilde{u}_1^{(s)}|_{\eta=\infty} = -\frac{3}{8}\left\{(1-\mathrm{i})\,U^*\frac{\mathrm{d}U}{\mathrm{d}x} + (1+\mathrm{i})\,U\frac{\mathrm{d}U^*}{\mathrm{d}x}\right\}$$
$$= \tfrac{3}{2}(1-\lambda^2)\sin 2x + 3\lambda. \quad (13)$$

This velocity 'drives' the steady streaming outside the Stokes layer; it is shown for various values of λ in figure 2, where we note $u_s(x+\pi) = u_s(x)$, and agrees in the two limiting cases $\lambda = 0$, 1 with earlier work, Riley (1971), Davidson & Riley (1972).

To derive an equation for the steady streaming at $O(\epsilon)$ outside the Stokes layer is not an entirely straightforward matter. For uni-directional fluctuations, corresponding to $\lambda = 0$, Riley (1967) has shown, by considering terms up to $O(\epsilon^3)$ in (1), that the steady streaming satisfies the full Navier–Stokes equations for steady flow with R_s as Reynolds number. Using similar techniques, Riley (1992) has generalized this to obtain, for the steady streaming, the equation

$$(1/R_s)\nabla^2 \zeta_1^{(s)} - (\boldsymbol{v}_L^{(s)} \cdot \nabla)\zeta_1^{(s)} = 0. \tag{14}$$

In (14), $\zeta_1^{(s)}$ is the vorticity such that $\nabla \wedge \boldsymbol{v}_1^{(s)} = (0, 0, -\nabla^2 \psi_1^{(s)}) = (0, 0, \zeta_1^{(s)})$ and $\boldsymbol{v}_L^{(s)} = \boldsymbol{v}_1^{(s)} + \boldsymbol{v}_d$ where

$$\boldsymbol{v}_d = \left\langle \left(\left(\int^t \boldsymbol{v}_0 \, dt \cdot \nabla \right) \boldsymbol{v}_0 \right\rangle, \tag{15}$$

is the Stokes drift velocity. Equation (14) shows a balance between diffusion and convection of the time-independent vorticity $\zeta_1^{(s)}$, but with the Lagrangian mean velocity $\boldsymbol{v}_L^{(s)}$ providing the convective effect. This result is not unexpected since the secondary velocity (15) is present even in an inviscid fluid; see also Lighthill (1956). Equation (14) has to be solved subject to $u_1^{(s)} = u_s$ on $y = 0$, with u_s as in (13), and the steady streaming effect decaying to zero at large distances from the cylinder. Lighthill (1978) has remarked that all worthwhile, i.e. concentrated, streaming motions take place with $R_s \gg 1$, and it is in this large-Reynolds-number limit that we consider solutions of (14). The outer solution itself then includes a boundary-layer structure. We introduce the classical boundary-layer variables $\bar{y} = R_s^{\frac{1}{2}} y$, $\bar{v}_1^{(s)} = R_s^{\frac{1}{2}} v_1^{(s)}$, and we note that in this thin outer boundary layer the drift velocity (15) takes the particularly simple form $\boldsymbol{v}_d = (2\lambda, 0)$, and (14) becomes

$$(\boldsymbol{v}_L^{(s)} \cdot \nabla)\, \zeta_1^{(s)} = \frac{\partial^2 \zeta_1^{(s)}}{\partial \bar{y}^2}, \quad \text{with } \boldsymbol{v}_L = (u_1^{(s)} + 2\lambda, \bar{v}_1^{(s)}). \tag{16}$$

Now, since in the boundary-layer limit $\zeta_1^{(s)} = -\partial u_1^{(s)}/\partial \bar{y}$, we may integrate (16) once to give the equation which must be solved as

$$(u_1^{(s)} + 2\lambda) \frac{\partial u_1^{(s)}}{\partial x} + \bar{v}_1^{(s)} \frac{\partial u_1^{(s)}}{\partial \bar{y}} = \frac{\partial^2 u_1^{(s)}}{\partial \bar{y}^2}, \tag{17}$$

together with

$$(\partial u_1^{(s)}/\partial x) + (\partial \bar{v}_1^{(s)}/\partial \bar{y}) = 0. \tag{18}$$

The boundary conditions for (17), (18) require periodicity so that

$$u_1^{(s)}(x + 2\pi) = u_1^{(s)}(x).$$

In addition we have conditions at $\bar{y} = 0$, to match with the Stokes-layer solution, namely

$$u_1^{(s)} = u_s = \tfrac{3}{2}(1 - \lambda^2) \sin 2x + 3\lambda, \quad \bar{v}_1^{(s)} = 0; \tag{19a, b}$$

and to complete the specification of the problem in this outer boundary layer we require a condition as $\bar{y} \to \infty$. Since the condition (19*a*) gives a circulation about the cylinder at $\bar{y} = 0$, when $\lambda = 0$, we cannot ignore the possibility that the solution in the outer boundary layer matches with a potential flow solution. Such a solution will be that for a potential line vortex, which requires

$$u_1^{(s)} \to \gamma/2\pi \quad \text{as } \bar{y} \to \infty. \tag{20}$$

This corresponds to $v_\theta = -\gamma/2\pi r$ outside the boundary layer, where the circulation γ is, as yet, unknown. The fact that $u_1^{(s)}$ is constant at the edge of the boundary layer justifies the omission of any pressure-gradient term in (17). We discuss the solution of (17)–(20) in the next section.

4. Results and discussion

The two limiting cases $\lambda = 0, 1$ have been discussed previously. For the case $\lambda = 1$, Riley (1971) has pointed out that the solution is, trivially, $u_1^{(s)} = \text{const}$, $\bar{v}_1^{(s)} \equiv 0$, so that $\gamma = 6\pi$. The case $\lambda = 0$ is more complex. Stuart (1966) conjectured that the boundary layers in the outer flow would collide at $x = \frac{1}{2}\pi, \frac{3}{2}\pi$ with jet-like flows erupting from the cylinder surface. A numerical solution of (17)–(20) with $\lambda = \gamma = 0$ has been obtained by Davidson & Riley (1972) which, together with experiment, upholds the conjecture. Calculations by Haddon & Riley (1979), based upon the steady Navier–Stokes equations, add further confirmation. In the present case, then, we may conjecture that for $\lambda_c < \lambda \leqslant 1$ the structure of the outer boundary layer is such that the fluid within it performs a circulatory motion around the cylinder. But there is a critical value, λ_c, such that for $0 \leqslant \lambda \leqslant \lambda_c$ fluid erupts from the boundary layer, in a jet-like manner, into the main body of fluid. For those values of λ, the model of the flow we have proposed in §3, embodied in (17)–(20) will not be valid.

To investigate the situation further we write, in (17)–(20), $u_1^{(s)} = -2\lambda + \bar{u}_1^{(s)}$ so that we now have

$$\bar{u}_1^{(s)} \frac{\partial \bar{u}_1^{(s)}}{\partial x} + \bar{v}_1^{(s)} \frac{\partial \bar{u}_1^{(s)}}{\partial \bar{y}} = \frac{\partial^2 \bar{u}_1^{(s)}}{\partial \bar{y}^2}, \quad \frac{\partial \bar{u}_1^{(s)}}{\partial x} + \frac{\partial \bar{v}_1^{(s)}}{\partial \bar{y}} = 0, \tag{21a, b}$$

with

$$\left. \begin{aligned} \bar{u}_1^{(s)} &= \tfrac{3}{2}(1-\lambda^2)\sin 2x + 5\lambda = \bar{u}_s, \text{ say}, \quad \bar{v}_1^{(s)} = 0 \quad \text{on } \bar{y} = 0; \\ \bar{u}_1^{(s)} &= \gamma/2\pi + 2\lambda = \bar{u}_\infty, \text{ say}, \quad \text{as } \bar{y} \to \infty. \end{aligned} \right\} \tag{22a, b}$$

The problem posed by (21), (22) may now be recognized as the 'Batchelor sleeve problem', and it may be readily shown (Batchelor 1956), that the relationship between $\bar{u}_s$ and $\bar{u}_\infty$ is simply

$$\int_0^{2\pi} \bar{u}_s^2 \, dx = \int_0^{2\pi} \bar{u}_\infty^2 \, dx, \tag{23}$$

which gives

$$\gamma/2\pi = -2\lambda + \{\tfrac{9}{8}(1-\lambda^2)^2 + 25\lambda^2\}^{\frac{1}{2}}. \tag{24}$$

The result (24) holds only when $\bar{u}_1^{(s)} > 0$ which, in the absence of any pressure gradient term in (21a) implies $\bar{u}_s > 0$, and this, in turn, from (22a) implies $\lambda \geqslant \lambda_0 \approx 0.28$. For values $\lambda_c < \lambda < \lambda_0$, the relationship between λ and γ must be found by numerical means. To enable that, we adopt the technique described by Riley (1981) for flows of the Batchelor type. Equations (21) are discretized in the x- and $\bar{y}$-directions using central differences and, starting with a profile that satisfies (22) but is otherwise arbitrary, the solution is advanced in the x-direction. After several sweeps over the interval $0 \leqslant x \leqslant 2\pi$ periodicity of the solution is achieved. All this is for given values of λ and γ. Periodicity does not determine γ, for a fixed value of λ; a more careful consideration of the matching between the boundary layer and the outer potential flow must be made. This is done by ensuring that the vorticity is zero at the outer edge of the boundary layer. If the computational domain is defined by $0 < \bar{y} < \bar{y}_\infty$, then for a given value of λ we calculate $S_\gamma = \Sigma |\partial \bar{u}_1^{(s)}/\partial y|_{\bar{y}=\bar{y}_\infty}$, where the summation takes place over all the grid points for $0 \leqslant x \leqslant 2\pi$. As γ varies we find that S_γ exhibits a well-defined minimum which we accept as the value of γ appropriate to the particular value of λ. With $\bar{y}_\infty = 10$, and $\delta x = \frac{1}{50}\pi$, $\delta \bar{y} = 0.1$, we find that $\min S_\gamma = O(10^{-4})$, which enables us to reproduce numerically the results given by (24), for $\lambda \geqslant \lambda_0$, to within $\frac{1}{2}\%$. We have used this technique for values of $\lambda < \lambda_0$ successfully down to $\lambda = 0.237$, when the main features of the flow we wish to

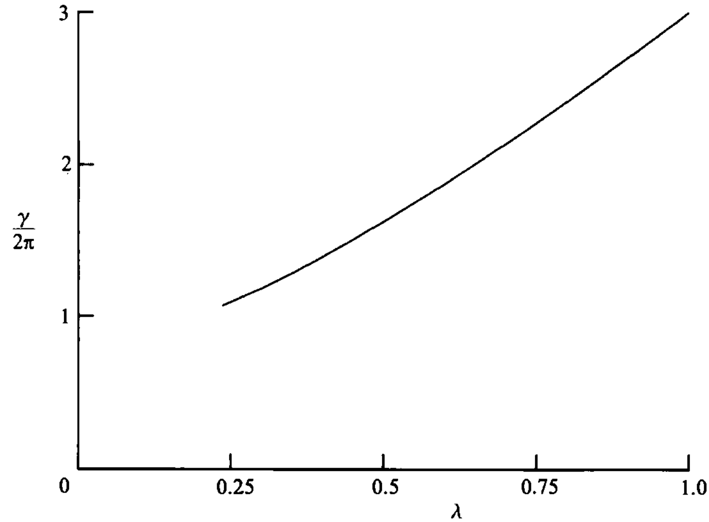

FIGURE 3. The circulation γ of the outer potential flow as a function of λ.

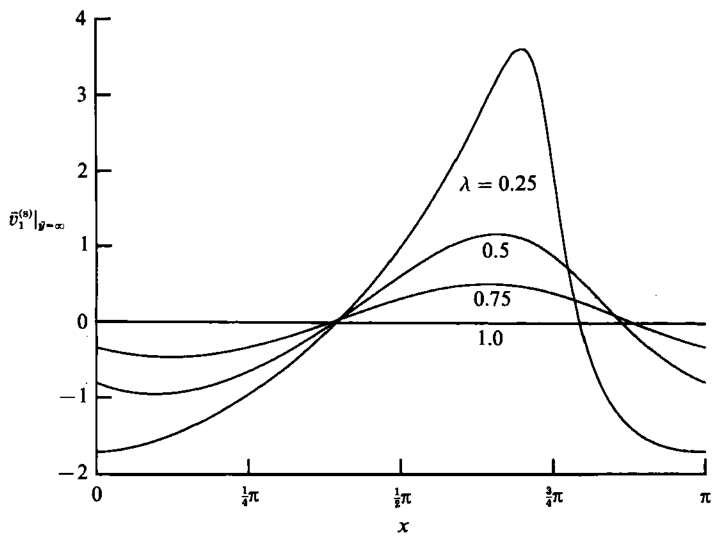

FIGURE 4. The normal component of velocity at the edge of the outer boundary layer, $\bar{v}_1^{(s)}|_{\bar{y}\to\infty}$, for various values of λ.

establish have clearly emerged. The failure of the method below this value may be attributed to an instability in our numerical scheme, which involves advancing the solution into regions where close to the boundary there is reversed flow, exacerbated by the incipient breakdown of the flow structure. The results we have obtained are shown in figures 3–5.

In figure 3 we present the variation of the circulation, associated with the outer potential flow, with λ. This is unexceptional, showing an expected monotonic decrease of γ with λ. In figure 4 we show the distribution of $\bar{v}_1^{(s)}|_{\bar{y}\to\infty}$ for $0 \leqslant x \leqslant \pi$ and various values of λ. As λ decreases below about 0.5 we see a dramatic increase in the maximum value of the normal velocity at the edge of the boundary layer. This is further illustrated in figure 5 which shows $\max \bar{v}_1^{(s)}|_{\bar{y}\to\infty}$ as a function of λ. The

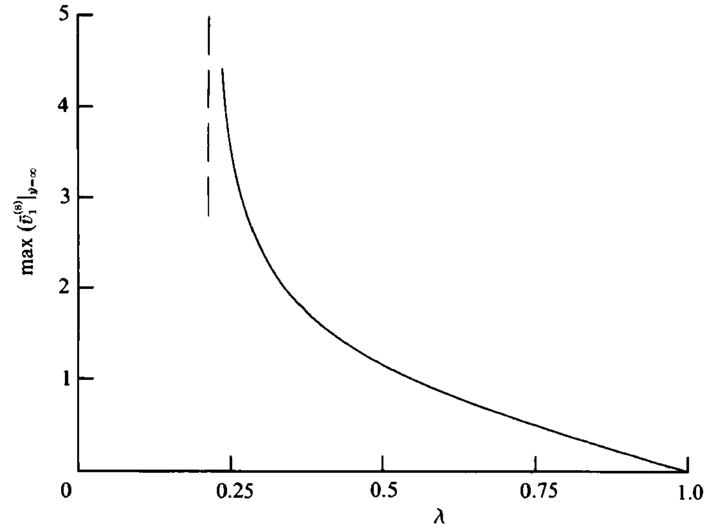

FIGURE 5. $\max \bar{v}_1^{(s)}|_{\bar{y}\to\infty}$ as a function of λ. The critical value $\lambda = \lambda_c \approx 0.213$ is shown as a broken line.

development of a singularity in this quantity is evident. Our results are consistent with $\bar{v}_1^{(s)}|_{\bar{y}\to\infty} \sim 0.684\ (\lambda - 0.213)^{-\frac{1}{2}}$ as this singular point is reached, although we have not been able to establish this result analytically. We interpret the breakdown at $\lambda = \lambda_c \approx 0.213$ as an eruption of fluid from the boundary layer which dramatically changes the flow structure proposed in §3. In the limiting case $\lambda = 0$, we have no net circulation, so $\gamma = 0$, and jets of fluid emerge symmetrically along $x = \frac{1}{2}\pi, \frac{3}{2}\pi$. For $0 < \lambda < \lambda_c$, although we again have a jet-like eruption of fluid from the boundary layers, it is not possible from the high-Reynolds number theory of §3 to infer the flow structure. Such an eruption will invalidate the arguments leading up to (17)–(20) and only a calculation of the steady streaming based upon (14) can resolve the flow behaviour.

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